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User Equilibrium in a Bottleneck under Multipeak Distribution of Preferred Arrival Time

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Abstract

This paper studies the pattern of departure times at a single bottleneck, under general heterogeneous preferred arrival times. It delivers three main outputs. Firstly, the existence of equilibrium is proven without the classical ”S-shape” assumption on the distribution of preferred arrival time i.e. that demand, represented by the flow rate of preferred arrival times, may only exceed bottleneck capacity on one peak interval. Secondly, a generic algorithm is given to solve the departure time choice equilibrium problem. Lastly, the graphical approach that pervades the algorithm provides insight in the structure of the queued periods, especially so by characterizing the critical instants at which the entry flow switches from a loading rate (over capacity) to an unloading one (under capacity) and vice versa. Numerical illustration is given.

Keywords

Traffic equilibrium. Bottleneck model. Departure time choice. Heterogeneous demand. Schedule delay.

Manuscript Text

1. INTRODUCTION

Transportation planners have long known how to determine equilibrium between travel demand and supply in a static framework. Yet, compared to dynamic models, static models tend not only to overestimate the traffic loads on major links in peak periods, but also to underestimate the travel time along them! It is much better in such cases to use a dynamic equilibrium model, featuring both the dynamics of traffic phenomena and congestion, and the users' choices of departure time. This is particularly true in the urban setting because the network users are able to adapt themselves to a time-varying quality of service by adjusting their departure time, by leaving earlier or later than initially planned so as to trade travel cost against the delay cost of a time lag at the destination between their actual and target arrival times.

The seminal paper on trip scheduling is due to Vickrey (*1*), who considered a fixed number of commuters traveling from an origin to a destination by a single route where congestion occurs at a bottleneck; each user being a microeconomic agent minimizing a cost function that involves travel time as well as schedule delay. In the simplest version of the model, Vickrey considered homogeneous users that have same preferred arrival time and same cost function.

Many extensions of the model have been provided in the literature, with focus on user heterogeneity. That pertaining to preferred arrival times has been treated by Hendrickson and Kocur (*2*) with no solution algorithm. Heterogeneity pertaining to the costs of travel time and of schedule delay has been addressed by e.g. Van Der Zijpp and Koolstra (*3*), Arnott *et al* (*4*). Other extensions include the modeling of stochastic demand and capacity, multiple routes or elastic demand – for review see Arnott *et al* (*5*).

The known results about the equilibrium pattern of departure times can be summarized as follows. When the preferred arrival time is common to all users, a single congestion period emerges with queue at bottleneck first increasing to a maximum and then vanishing (*4*). Smith (*6*) and Daganzo (*7*) showed that this simple departure pattern holds for a distribution of preferred arrival times, under the so-called "S-shape" assumption of a unique peak period, i.e. a single interval on which the density of preferred arrival times exceeds the bottleneck capacity rate. However, in the case of a finite number of preferred arrival schedules and heterogeneous cost functions, Lindsey (*8*) and Van Der Zijpp and Koolstra (*3*) showed that the resulting departure pattern may be much more complex with possibly several congestion periods and multiple maxima in queuing time. Ramadurai *et al* (*9*) addressed a similar model to (*8*) in a game-theoric framework.

The purpose of this paper is to extend model of Smith and Daganzo to a general distribution of preferred arrival times. Indeed this induces a complex pattern of departure times, as in (*3*) and (*8*). The core principle in our analysis is to cast the distribution of departure times into a differential equation which involves the distribution of preferred arrival times, as mediated by bottleneck flowing, together with the costs of schedule delay and travel time. The differential equation characteristic of equilibrium also inspires a solution algorithm, which consists in searching for candidate initial instants of queued periods.

The paper is organized into four main parts and a conclusion. First, Section 2 states the modeling assumptions and provides intuitive reasoning into the structure of the equilibrium pattern. Then, in Section 3 the characteristic differential equation is obtained by mathematical analysis of the optimality conditions. Next, Section 4 states the solution algorithm and provides a theorem of existence of a departure time equilibrium under general distribution of preferred arrival times. Section 5 is devoted to numerical illustration. Lastly, Section 6 gives some concluding comments.

2. THE MODEL

Consider a single origin-destination pair connected by a single route, and a set of N users with heterogeneous preferred arrival times. In a game-theoretic perspective, every user is modeled as a microeconomic agent seeking unilaterally to minimize a travel cost function by adjusting his departure time h. His choice behavior involves a cost function of the travel time $w(h)$ at h; the distribution of individual choices gives rise to a distribution of departure times which makes a cumulated trip volume at the entrance of the route, which may be called the demand. In turn the macroscopic entry trip volume, denoted as $X_+(h)$, determines the route travel time $w(h)$ on the basis of queuing dynamics. The travel time function w represents the supply state. The demand function linking X_+ to w, and the supply function linking w to X_+, make up a circle of dependency, typical of an equilibrium problem between supply and demand.

This section is purported to specify the assumptions first on the supply side, then on the demand side, so as to state the equilibrium problem in a formal way.

The following notations will be used:

- H_+, H_- and H_p respectively are the domains of departure, arrival and preferred times. Without going into the details, let us assume that these are sufficiently large intervals so that no departure and arrival takes place out of them.
- X_+ is a distribution of departure time over H_+ i.e. $X_+(h)$ represents the number of users having departed before h hence also the cumulated trip volume. X_+ is assumed to be continuous and differentiable nearly everywhere, with time derivative $x_+(h)$ to be interpreted as the flow rate of departing users at h. A last requirement on X_+ is that at a maximum instant $h_{\max}$, $X_+(h_{\max}) = N$ the total number of users.
- K the bottleneck capacity, a flow rate.
- w defined on H_+ is a travel time function assumed to be continuous and differentiable nearly everywhere.
- W the function that maps a distribution X_+ to a travel time function w.
- The differentiation of function f with respect to instant h is denoted as $\dot{f}$.

2.1 Transport supply - Flowing model.

Let us first consider the derivation of travel time function w from departure time distribution X_+. Travel along the route is assumed uncongested except perhaps at a single bottleneck of deterministic capacity K. If the entry flow coming in bottleneck has rate in excess of K, then a waiting queue develops where users wait to leave queue according to a First In – First Out discipline. Let us define the travel time function w by the following relationship, in which $Q(h)$ denotes the number of queued users at h in the bottleneck, and t_0 is the free flow travel time:

$$w(h) = t_0 + \frac{Q(h)}{K} \tag{1}$$

where Q stems from the following differential equation :

$$\dot{Q}(h) = \begin{cases} x_+(h) - K & \text{if } Q(h) \neq 0 \text{ or } x_+(h) - K > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

When X_+ is continuous, the resulting travel time w is well defined and is continuous and differentiable nearly everywhere. Without loss of generality, we assume that $t_0 = 0$ thus making w to stand for waiting time.

The flowing model is represented in a compact way by the following notation:

$$w = \mathrm{W}(X_+) \tag{3}$$

2.2 Demand side

User behavior. Every user is characterized by a preferred arrival time $\eta \in \mathrm{H}_p$ and a travel cost function representing a trade-off between a travel time and a schedule delay, defined as the arithmetical time lag between the actual arrival time $\hbar$ and η. Given travel time function w, the cost g to a user with preferred arrival time η upon departing at h is defined as:

$$g^{[w]}(h,\eta) = \alpha w(h) + \mathrm{D}\big(h + w(h) - \eta\big) \tag{4}$$

where D is the *schedule delay cost function* and α the *trade off between cost and time* also referred to as the value of time (to the user). Let also:

Assumption 1, on Cost of Schedule Delay

a) D *is continuous.*

b) D *is differentiable on* $\Re \setminus \{0\}$ *with derivative* D_ℓ

c) D *is convex.*

d) D *achieves a minimum at* 0 *and* $\mathrm{D}(0) = 0$.

These are standard assumptions, e.g. (*4*), (*8*) and yield a cost of schedule delay that increases with the lag between actual and preferred arrival time. Assumptions 1c and 1d make D to decrease on $\Re^-$ and increase on $\Re^+$.

Each user is an economic agent modeled as a rational decision-maker with perfect information: he chooses his departure time so as to minimize his cost function. Given his preferred arrival time η and the travel time function w, his choice of departure time amounts to the following mathematical program:

$$\min_h g^{[w]}(h,\eta) \tag{5}$$

The distribution of users. Consider now a set of N users with a same cost function g, but heterogeneous preferred arrival times. This is represented by a cumulative distribution X_p on H_p: $X_p(\eta)$ is the number of those users with preferred arrival time is less than η. The derivative of X_p, denoted as x_p, is defined almost everywhere and is readily interpreted as the flow rate of users with preferred arrival time η. From its definition, X_p is increasing and semi-continuous. Let also:

Assumption 2, on the Distribution of Preferred Arrival Time:

a) X_p *is continuous.*

b) $x_p > K$ *on a finite number of intervals.*

c) $x_p \neq K$ *almost everywhere*

Assumption 2b generalizes the S-shape assumption considered in (*2*), (*6*) and (*7*), which could be stated as "$x_p > K$ on a single interval". Those intervals are called *peak periods* as along each of them there are more users that would prefer to arrive than allowed by the route capacity. Intuitively a higher number of peak periods will give rise to a more complex distribution of departure time, with potentially several distinct queuing periods. Assumption 2a is purely technical, so is 2c which is required only to make precise the statement of the algorithms in Section 4.

The order of departure. In the literature, little consideration has been given to represent the departure choice decision of a continuous distribution of users. A natural approach is to introduce a departure choice function H mapping a user with preferred arrival time η to his chosen departure time h. Then distribution X_+ stems from:

$$X_+(h) = \int 1_{\{H(\eta)\le h\}}\, \mathrm{d}X_p(\eta)\,. \tag{6}$$

Yet, relation (6) is not convenient to handle. For the sake of analytical simplicity, let us assume:

Assumption 3, on Natural Order*. The departure choice function H is continuous and increasing.*

This implies that users depart in the order of increasing preferred arrival time, and hence is referred to as the natural order assumption. An obvious issue pertains to the existence of an equilibrium choice function which would not satisfy to a natural order. Daganzo (*7*) investigated the case with a strictly convex schedule delay costs function and showed that natural order is satisfied by measurable functions of equilibrium choice of departure time. However, this does not extend to barely convex functions, as showed in (*5*).

Under the natural order assumption, equation (6) becomes:

$$X_+ = X_p \circ H^{-1}. \tag{7}$$

For an increasing function F such as X or H, our definition of its reciprocal function F^{-1} is as follows:

$$F^{-1}(x) \equiv \inf\{h : F(h) > x\}.$$

2.3 Stating the problem of User Equilibrium

Each user tries to minimize his cost function under perfect information. By definition, the user equilibrium (UE) is a situation where no user can reduce his cost by unilaterally changing his decision, here of departure time.

A natural statement of the problem is:

Definition 1, User equilibrium based on departure time function*. Find an increasing function $H(\cdot)$ such that, letting $X_+ \equiv X_p \circ H^{-1}$:*

$$g^{[w]}(H(\eta),\eta) \quad \leq \quad g^{[w]}(h',\eta) \text{ for almost every } \eta \in \mathrm{H}_p,\, h' \in \mathrm{H}_+, \tag{8a}$$

$$w = \mathrm{W}(X_+). \tag{8b}$$

The associated distribution of departure times stems from natural order. Eqn (8a) expresses the impossibility for any user to improve on his departure time decision; Eqn (8b) is the flowing equation.

Let us provide a simpler alternative formulation:

Definition 2, User equilibrium based on departure time distribution*. Find an increasing function $X_+(\cdot)$ such that, letting $H_p \equiv X_p^{-1} \circ X_+$:*

$$g^{[w]}(h, H_p(h)) \quad \leq \quad g^{[w]}(h', H_p(h)) \text{ for almost every } h,\, h' \in \mathrm{H}_+, \tag{9a}$$

$$w = \mathrm{W}(X_+). \tag{9b}$$

In (9a) the optimality condition is expressed by enumerating the users in order of departure time, whereas in (8a) each user is labeled by his preferred arrival time. The relationship between the two arises from the fact that, in natural order, the n-th user to depart is also the n-th user in the order of preferred arrival time.

The two problems are equivalent in the following way.

Proposition 1, Equivalency of equilibrium statements. *(i) A solution X_+ of (9) yields a solution $H \equiv X_+^{-1} \circ X_p$ of (8). (ii) Conversely, if H is a solution of (8) then $X_+ \equiv X_p \circ H^{-1}$ is a solution of (9).*

Proof. (i) Assume that X_+ is a solution to (9) and consider $H \equiv X_+^{-1} \circ X_p$. Then H is defined, an increasing function of *h* as the composition of two increasing functions, with associated departure distribution X_+. Consider $\eta \in \mathrm{H}_p$ and apply (9a) to $h = H(\eta)$: then for all $h' \in \mathrm{H}_+$ it holds that $g^{[w]}(H(\eta),\eta) \leq g^{[w]}(h',\eta)$ hence (8a). (ii) Same argument in reverse order.

This enables us to study the equilibrium by focusing on X_+ rather than H. In the sequel, we address the UE problem in departure time distribution.

3. PROPERTIES OF EQUILIBRIUM DEPARTURE TIME DISTRIBUTION

In this section, necessary conditions are derived on an allegedly optimal pattern X_+ from the optimality equation (9). Then these conditions are shown to be also sufficient. This line of attack had already been taken by Smith (*6*), but in the specific case of an S-shape distribution of preferred arrival time.

3.1 On queued and peak periods

Assuming that X_+ is a solution of the UE problem, let us consider $w = \mathrm{W}(X_+)$. As w is continuous, the sets of h such that $\{w = 0\}$ [resp. $\{w > 0\}$] are countable unions of closed [resp. open] intervals. We refer to those intervals as *unqueued* [resp. *queued*] periods.

Consider first an unqueued period U: users departing during U incur only a cost of schedule delay. Thus, it is optimal for a user with preferred arrival time η to choose h interior to U if and only if he has $h = \eta$. Otherwise he could lower his cost by marginally changing h towards η. Then at equilibrium $H_p = Id$ on U and $x_+ = x_p$.
Now consider a queued period Q. As w is continuous, non negative and is zero at the endpoints of the period interval, it has a least one maximum value and possibly minima. The general pattern of travel time is therefore expected to be a sequence of increasing then decreasing sub-periods.

This gives us a crucial insight into the structure of an equilibrium state. First, whenever there is no queue, users arrive (and depart) at their preferred arrival time and thus incur no cost. Second, the peak periods defined above (at $x_p > K$), play an important role in the problem: as unqueued departure flow is equal to scheduled flow at arrival, an unqueued period cannot intersect a peak period except perhaps at isolated points (since $w = 0$ cannot be sustained when $x_+ > K$). Therefore, the maximum number of queued periods is bounded by the number of peak periods; whereas the number of unqueued periods is limited to one plus that bound.

To sum up, we have highlighted two important features of H_+ and H_p under an equilibrium distribution. The set of departure times is divided into alternated periods of unqueued and queued states. Provided that H_+ be "large enough", the first and last periods should be unqueued. To state this principle explicitly, we denote $Q_1 =]q_0, q_1[, \ldots, Q_{2n_q+1}$ the sequence of unqueued and queued periods, q_{2k+1} and q_{2k+2} being *transition instants* from an unqueued period to the next queued period, and from queued to unqueued, respectively. Similarly, we denote by $P_1 =]p_0, p_1[, \ldots, P_{2n_p+1}$ the sequence of successive peaks (when $x_p > K$) and off peak (when $x_p < K$) periods in H_p.

3.2 Necessary conditions

Given a solution X_+ of the UE problem (9), consider the associated functions of travel time $w = \mathrm{W}(X_+)$, preferred time $H_p = X_+^{-1} \circ X_p$ and cost g (the superscript w is omitted for the sake of legibility). Our aim is to turn the optimality conditions on the basis of g into conditions on X_+ by means of the flowing equation. To do so, the two states of unqueued versus queued traffic must be addressed as distinct cases. About unqueued periods, we already established that

$$x_+ = x_p\,, \tag{10}$$

and it holds that $w(h) = 0$ and $H_p(h) = h$. Then $h = X_+^{-1} \circ X_p(h)$ and $X_+(h) = X_p(h)$. This applies notably to each instant q_i of transition between queued and unqueued state, yielding that

$$X_+(q_i) = X_p(q_i) \text{ for any } i \in \{0,1..2n_q\} \tag{11}$$

About a queued period Q, for a given departure instant h in H_+, with $H_p(h)$ the preferred arrival time of the users departing at h, consider the function $g^{(h)} : h' \mapsto g(h', H_p(h))$. As the functions $h' \mapsto w(h)$ and $h' \mapsto \mathrm{D}(h' + w(h') - H_p(h))$ are differentiable a.e. on Q, so is $g^{(h)}$. Denoting $\dot{g}^{(h)}$ its derivative, for almost every h it must hold that $\dot{g}^{(h)}(h') = 0$.

Yet as D is differentiable on $\Re \setminus \{0\}$, whenever $h' + w(h') - H_p(h) \neq 0$ $\dot{g}^{(h)}$ is

$$\dot{g}^{(h)}(h') = \alpha \dot{w}(h') + \mathrm{D}_\ell(h' + w(h') - H_p(h))(1 + \dot{w}(h')) \tag{12}$$

Eqn 12 is easily extended a.e. on Q by defining $\mathrm{D}_\ell(0) \equiv 0$. For almost every h in Q, we thus have:

$$\alpha \dot{w}(h') + \mathrm{D}_\ell(h' + w(h') - H_p(h))(1 + \dot{w}(h')) = 0 \tag{13}$$

Introducing the flowing equation (3), we get that:

$$x_+ = K.\frac{\alpha}{\mathrm{D}_\ell(\ell) + \alpha} \tag{14}$$

where $\ell(h) \equiv h + w(h) - H_p(h)$ is the arrival time lag of the user departing at h.

Eqn (14) has two remarkable features. First $x_+ < K$ whenever $\ell > 0$ and $x_+ > K$ when $\ell < 0$. Yet, ℓ can be interpreted as the schedule delay incurred by a user departing at h. Consequently, each queued period can be divided in early sub-periods when users depart early (that is, depart at a time yielding arrival earlier than preferred *ex-ante*), during which the entry flow rate is beyond capacity and the queue builds up; and late sub-periods when users depart late, during which the entry flow rate is under capacity and the queue diminishes. Second, (14) can be stated as a differential equation in X_+ over $Q_i =]q_{i-1}; q_i[$. Indeed, according to the flowing equation (3) we have $\dot{w} = (x_+ - K)/K$ on Q_i, so by integrating over $]q_{i-1}; h[$:

$$w(h)+h=q_{i-1}+\frac{X_+(h)-X_+(q_{i-1})}{K} \tag{15}$$

Taking the definition of $H_p = X_p^{-1}\circ X_+$, the arrival time lag ℓ can now be expressed as a function of X_+, so that (14) yields the following differential equation in $X_+ \equiv \int x_+ \,\mathrm{d}h$:

$$\frac{\mathrm{d}X_+}{\mathrm{d}h}=K.\frac{\alpha}{\alpha+\mathrm{D}_\ell\left(q_{i-1}+\frac{X_+(h)-X_+(q_{i-1})}{K}-X_p^{-1}\circ X_+(h)\right)} \tag{16}$$

To sum up, we have shown that the equilibrium departure time distribution satisfies the differential equations (10) and (16) respectively on unqueued and queued intervals. Successive integrations of these equations along the Q_i periods with appropriate initial condition coming from the previous period yields the equilibrium departure time distribution, provided that the Q_i periods are given.

3.3 Necessary and Sufficient Conditions

Let us now demonstrate that the necessary conditions are also sufficient conditions, owing to the following property:

Proposition 2. *Let X_+ be a departure time distribution with associated sequence Q_i of unqueued and queued periods. Then X_+ is an equilibrium solution state if and only if it satisfies (14) and (11) on Q_{2i} and (10) on Q_{2i+1}.*

Proof. Having demonstrated the "only if" part in the previous subsection, let us tackle the "if" part by taking a departure time distribution X_+ with associated distributions $w=\mathrm{W}(X_+)$ and $H_p = X_p^{-1}\circ X_+$ of travel time and preferred time, respectively. It is assumed that X_+ satisfies (14), (11) on Q_{2i} and (10) on Q_{2i+1}. Let us fix any h in H_+ and consider the function $g^{(h)}: h' \mapsto g(h', H_p(h))$. Our aim is to show that $g^{(h)}$ admits a global minimum at $h'=h$. From its definition $g^{(h)}$ is continuous and differentiable almost everywhere, with derivative $\dot{g}^{(h)}$ given by (12).

Since H_p is an increasing function (as composition of two increasing functions), as is D_ℓ because of the convexity of D, $\dot{g}^{(h)}$ is a decreasing function of h': around point $h'=h$ we have that:

$$\dot{g}^{(h)}(h') \gtrless \dot{g}^{(h)}(h) \quad \text{if} \quad h' \lessgtr h \tag{17}$$

Yet $\dot{g}^{(h)}(h)=\alpha\dot{w}(h)+\mathrm{D}_\ell(h+w(h)-H_p(h)).(\dot{w}(h)+1)$ is zero almost everywhere on the basis of either (14) in a queued state or (10) in an unqueued state. This it holds that for almost every h, $h' \in \mathrm{H}_+$,

$$\dot{g}^{(h)}(h') \gtrless 0 \quad \text{if} \quad h' \lessgtr h \tag{18}$$

which means that $h'=h$ is the unique minimum of function $g^{(h)}$. Thus X_+ satisfies the optimality condition (9a), as well as (9b) by assumption.

3.4 Graphical interpretation of the NSC under V-shape schedule delay costs

From here it is assumed that D has the simple, V-shape form:

$$\mathrm{D}(h+w-\eta)=\beta(h+w-\eta)^{+}+\gamma(\eta-h+w)^{+} \tag{19}$$

where β [resp. γ] are the *marginal cost of arriving early [resp. late]* with respect to the preferred time η and $()^{+}$ denotes the positive part. Under this V-shape form, equation (14) can be restated in the following simple way:

$$x_{+}(h)=\begin{cases} x_{+}^{E}\equiv K\alpha/(\alpha-\beta) & \text{if } h+w(h)<H_{p}(h) \\ x_{+}^{L}\equiv K\alpha/(\alpha+\gamma) & \text{if } h+w(h)>H_{p}(h) \end{cases} \tag{20}$$

Therefore only two departure flows are admissible in a queued period, one made of users planning to arrive early regarding their preferred time and the other of users planning to arrive late. These are denoted by x_{+}^{E} and x_{+}^{L}, respectively, E and L standing for early and late. From their definition $x_{+}^{E}>K$ and $x_{+}^{L}<K$.

Let us now use the cumulated volume representation to comment the conditions on X_{+}. Figure 1 depicts X_{+}, H_{p} and $X_{-}=X_{+}(h+w(h))$, the arrival time distribution.

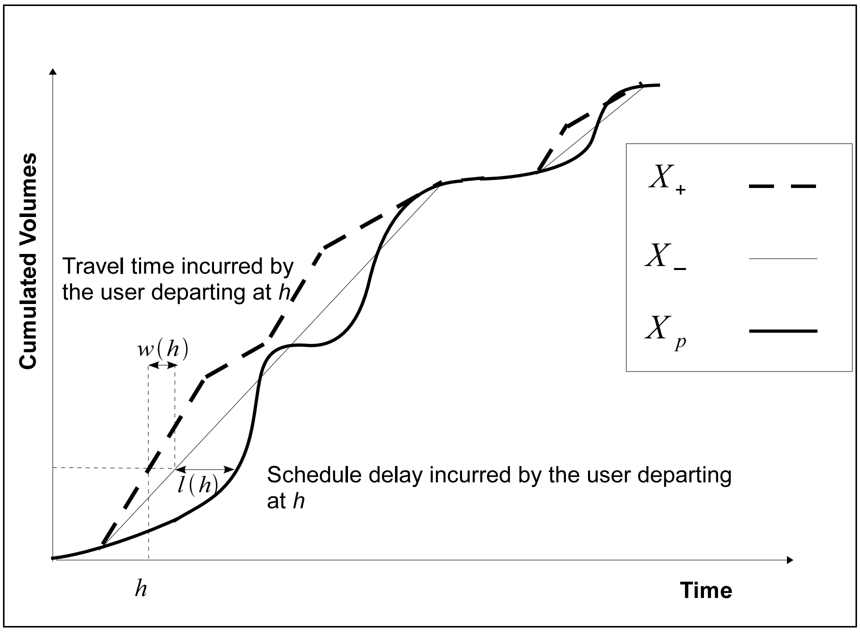


FIGURE 1 Cumulated volume representation of an equilibrium situation

First, note that X_{-} can be easily deduced from the sequence of the Q_i. Indeed, according to the simple flowing model, the exit flow rate is the capacity K on a queued period and so X_{-} has slope K; out of queued periods X_{+} simply coincides with X_{-} and X_{p}. Second, in Figure 1 one can read w and ℓ from the horizontal distance between respectively the graphs of X_{-} and X_{+}, and those of X_{-} and X_{p}. Moreover the intersection points between the graphs of X_{-} and X_{p} divide each queued period Q into early and late intervals regarding the preferred arrival time. The transition instants between two successive periods make *critical times at arrival*, denoted as $\hbar_i$. Such instants on a period $Q=[q_b, q_e]$ are the solutions of the equation:

$$K.(\hbar - q_b) = X_p(\hbar) - X_p(q_b) \tag{21}$$

Clearly there cannot be more than one $\hbar_i$ per peak or off peak period, and their total number over a queued period must be odd.

To each critical time at arrival $\hbar_i$ let us associate the corresponding departure time h_i, so that they are related by the equation:

$$\hbar_i = h_i + w(h_i) \tag{22}$$

The critical times at departure also divide each queued period Q_{2i} in intervals of earliness or lateness regarding the departure, i.e. in periods where users depart at a time such that they arrive early or late. Those instants correspond to a switch in the departure flow from x_+^E to x_+^L or conversely.

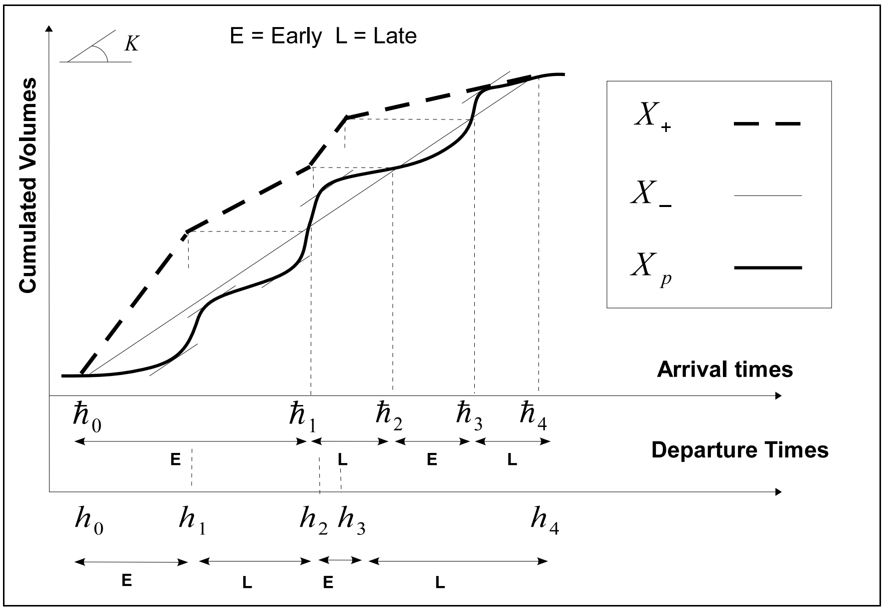


FIGURE 2 Critical times at arrival and at departure

4. UE ALGORITHM UNDER V-SHAPED COST OF SCHEDULE DELAY

This section provides an algorithm to compute the equilibrium departure time distribution based on the properties established previously. The objective of the algorithm is to build the distribution of departure time by determining the queued periods Q_{2k}. The principle is that, given the beginning of a queued period, both X_+ and w are easy to compute by integrating equations (14) and (2) and stopping when $w=0$: thus the main unknown variable is the initial instant of a queued period, and the algorithm is purported to test candidate initial instants.

Two questions arise about a candidate initial instant. First, will the associated queued period induce an equilibrium state? Then, how to search for all queued periods in such a way as to delimit precisely each of them? Both issues are addressed in an integrated way, by progressive identification of the successive queued periods. A criterion is provided that both guarantees the current queued period to be correct and ensures that the search for the next queued period should focus on later instants.

We shall first present an algorithm for testing a candidate initial instant $\hat{\hbar}_0$, then expose the full computation method and next give the proof of convergence. Lastly, based on the algorithm termination we derive the following existence result:

Theorem 1, Existence of equilibrium. *The user equilibrium problem with general preferred arrival time distribution and V-shaped cost of schedule delay admits at least one solution.*

4.1 Testing a candidate initial instant of a queued period

Assuming that a sequence of queued periods has been identified up to time h_b, our aim is to identify the initial instant $\hat{\hbar}_0$ of the next queued period, prior to the beginning of the next peak period.

The algorithm is as follows. First equation (21) is solved on $[\hat{\hbar}_0,+\infty[$, yielding a sequence of solutions $\hat{\hbar}_i$, which is referred to as the sequence of intersection times at arrival. Then the sequences $(\hat{h}_i)$ and $(\hat{w}_i)$ are derived in a recursive way, by setting initial value to $\hat{h}_0 = \hat{\hbar}_0$ and $\hat{w}_0 = 0$ and by using the following, recursive formulae:

$$x_+^i(\hat{h}_{i+1} - \hat{h}_i) := K(\hat{\hbar}_{i+1} - \hat{\hbar}_i) \text{ with } x_+^i = x_+^E \text{ if } i \text{ is even or } x_+^L \text{ if odd, and} \tag{23}$$

$$\hat{w}_i := \hat{\hbar}_i - \hat{h}_i \tag{24}$$

The sequences $(\hat{\hbar}_i)$, $(\hat{h}_i)$ and $(\hat{w}_i)$ are purely geometric constructions, as illustrated in Figure 3. Yet intuitively $(\hat{\hbar}_i)$ and $(\hat{h}_i)$ would correspond to the *i*-th critical times at arrival and departure derived from a given candidate $\hat{\hbar}_0$ and $(\hat{w}_i)$ to the corresponding waiting times. They define a candidate distribution $\hat{X}_+$ that *a priori* is not flow-consistent with the candidate arrival time distribution $\hat{X}_-$. Two unphysical phenomena may occur:

- "Travel time becomes negative": for some i, $\hat{h}_i < \hat{\hbar}_i$ or equivalently $\hat{w}_i < 0$. This typically corresponds to a situation where the candidate queued period started too early.
- "Queue does not vanish": for all i, $\hat{h}_i > \hat{\hbar}_i$ or equivalently $\hat{w}_i > 0$, which corresponds to a situation where the candidate queued period started too late.

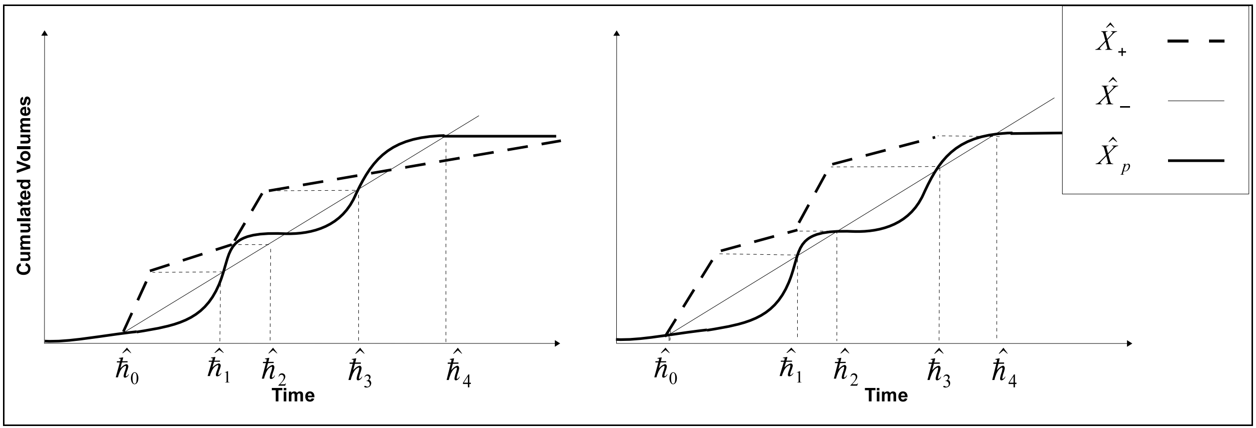


FIGURE 3 Testing a candidate initial instant

We claim that the sequence $\hat{w}_i$ allows us to assess the suitability of $\hat{\hbar}_0$ as initial instant of queuing in an equilibrium state. The intuition is as follows: assume that there exists k such that $\hat{w}_k = 0$ and $\hat{w}_i > 0$ for $i < k$. Then, by deriving X_+ from the sequence $(\hat{h}_i)_{i \le k}$, (14) and (16) hold on $Q = [q; \hat{h}_i]$ and Q indeed describes a queued period. Therefore, the condition "$\exists k$ such as $\hat{w}_k = 0$ and $\hat{w}_i \ge 0$ for $i < k$" is a necessary condition for $\hat{\hbar}_0$. Yet, it will be seen later on to be too weak for sufficiency; the appropriate criterion is in fact "$\exists k$ such that $\hat{w}_k = 0$ and $\hat{w}_i \ge 0$ for all i" or equivalently "$\min_i \hat{w}_i = 0$". Intuitively, this guarantees that the candidate queued period "leaves enough space" for the subsequent ones.

The algorithm is stated below in explicit pseudo-code.

Algorithm 1: `QTest` $(\hat{\hbar}_0)$

Outputs: h_e, $\min_i \hat{w}_i$

Set $\hat{w}_0$ **to** 0 **and** $\hat{h}_0$ **to** $\hat{\hbar}_0$

Solve $K.(\hbar - q) = X_p(\hbar) - X_p(q)$ on $[q, +\infty[$ and **Set** the n solutions **to** the sequence $(\hat{\hbar}_i)_{i=0..n-1}$ in increasing order.

For $i = 1..n-1$ **do**:

Set $\hat{h}_i := \hat{h}_{i-1} + K / [x_+(\hat{\hbar}_i - \hat{\hbar}_{i-1})]$ with x_+ equal to x_+^E if i is even or to x_+^L otherwise

Set $\hat{w}_i := \hat{\hbar}_i - \hat{h}_i$

End For

Set k **to** $\arg\min_i \hat{w}_i$ **and** h_e **to** h_k

4.2 Main algorithm

The general philosophy of our method is to find successively the queued periods in the UE departure time distribution, starting from the first peak period. Algorithm 2 consists in searching over an interval $[h_b, h_e]$ for the initial instant of a queued period, by testing candidate initial instants $\hat{\hbar}_0$ on the basis of Algorithm 1. The search method is a dichotomy process oriented by the sign of $\min_i \hat{w}_i$. Algorithm 3 uses Algorithm 2 repeatedly until all peak periods have been addressed; it returns the sequence of queued periods which fully determines X_+. The computation process is illustrated in Figure 4.

Algorithm 2: `findQueuedPeriod`($[h_b, h_e]$)

Outputs: $[q_b, q_e]$

Parameter ε a tolerance level

Ensure $x_p < H$ on $[h_b, h_e]$

Repeat

  Set $q_b := (h_e + h_b)/2$

  Set $\{q_e, \min\_w\}$ **to** QTest (q_b)

  If $\min\_w > 0$ **then Set** $h_e := q_b$

  else Set $h_b := q_b$

Until $|\min\_w| < \varepsilon$.

Algorithm 3: `equilibriumComputation`(H₊)

Outputs: (Q_{2k})

Set $k := 1$

Set h_e **to** initial instant of first peak period.

Set $h_b := \inf \mathrm{H}_+$

Repeat

  Set Q_{2k} **to** `findQueuedPeriod`($[h_b, h_e]$)

  Set $k := k + 1$

  Set $h_b := \sup Q_{2k}$

  Set h_e **to** initial instant of first peak period after Q_{2k}

Until there is no peak period after h_b

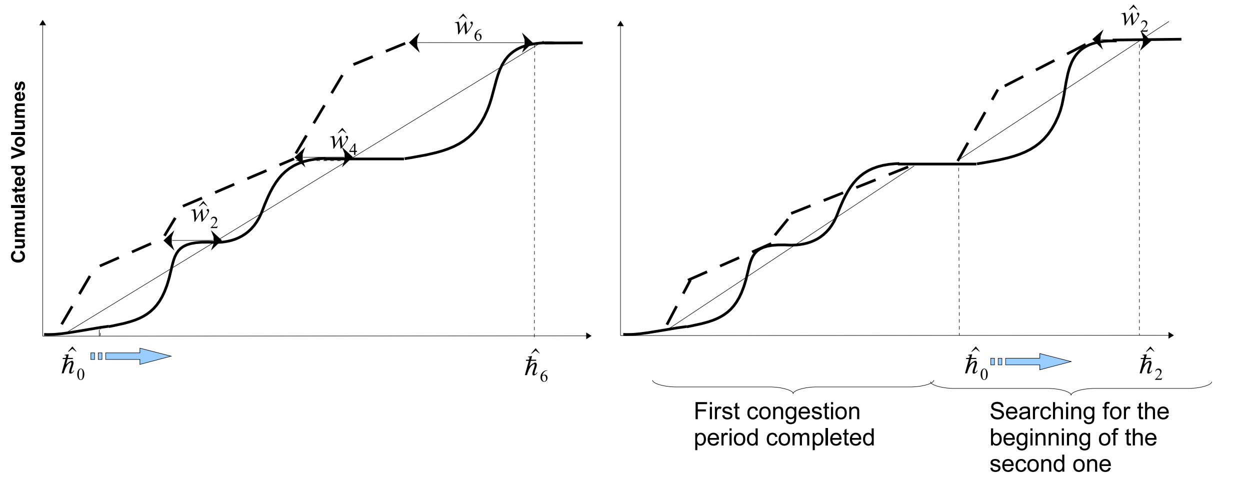


FIGURE 4 User equilibrium algorithm

4.3 Proofs of termination and existence theorem

Consider the functions $\hat{w}_i(h_0)$ defined by (23), (24) for each $h_0 \in [h_b , h_e]$ a given period. The following property is demonstrated in the next subsection.

Proposition 3. $W_{be}(h_0) \equiv \min_i \hat{w}_i(h_0)$ *is a continuous and decreasing function.*

This implies that the equation $W_{be} = 0$ has a solution at h_0 on $[h_b , h_e]$ if "$W_{be}(h_0) \geq 0$ and $W_{be}(h_0) \leq 0$". Then Algorithm 2 applied to an off-peak period with a subsequent peak and no queue inherited from previous peaks, hence $W_{be}(h_b) \geq 0$ and $W_{be}(h_e) \leq 0$, must terminate and yield a suitable $h_0 \in [h_b , h_e]$. Moreover, by progressive identification of the successive queuing periods in the equilibrium state, Algorithm 3 must terminate.

Let us finally address the issue of existence for an equilibrium departure time distribution. Consider the departure time X_+ computed from the outputs (Q_{2k}) of Algorithm 3 together with its associated distributions w and H_p of travel time and preferred time, respectively. Then for all k, $w \geq 0$ on Q_{2k} and $w = 0$ elsewhere. Moreover X_+ satisfies by construction (14) and (16) in the queued case and (10) in the unqueued case. The existence theorem then follows directly from Proposition 2.

4.4 Proof of Proposition 3

This subsection can be omitted without loss of continuity.

Consider an interval $[h_b, h_e]$ included in an off-peak period and denote $P_i = [p_{i-1}, p_i]$, $i = 1..2n$ the sequence of peak and off-peak periods after h_e. The proof proceeds in three steps. We shall first define for each i a function $\hat{\hbar}_i(h_0)$ on $[h_b, h_e]$ that takes its value in P_i. Second, some properties of these functions will be established. Third, we shall conclude about $\min_i \hat{w}_i$.

We shall make use of an auxiliary function as follows:

$$(h_0, \hbar) \mapsto \Delta(h_0, \hbar) \equiv K.(\hbar - h_0) - X_p(\hbar) + X_p(h_0)$$

Step I: Defining $\hat{\hbar}_i(h_0)$

For any h_0 in $[h_b, h_e]$ let us define $\hat{\hbar}_0, \ldots, \hat{\hbar}_{2n}$ by setting $\hat{\hbar}_0 := h_0$ and by using the following recursive rule. For any i from 1 to $2n$, try to solve the equation $\Delta(h_0, \hbar) = 0$ in $\hbar$ on P_i: if there is a solution $\hbar$ then set $\hat{\hbar}_i$ to $\hbar$, else set $\hbar_i$ to either p_i or p_{i-1} according to the following table of cases.

Case	$\Delta > 0$ on P_i	$\Delta < 0$ on P_i
i odd	p_i	p_{i-1}
i even	p_{i-1}	p_i

The derivation of a sequence $(\hat{\hbar}_i)$, illustrated in Figure 5, stands as an ad-hoc extension of formula (21) so as to address degeneracy in the number of queuing sub-periods: when several neighboring peak periods give rise to a common, queuing-dequeuing couple of sub-periods, then there is only one "true" critical time of maximal waiting, located in an off-peak period.

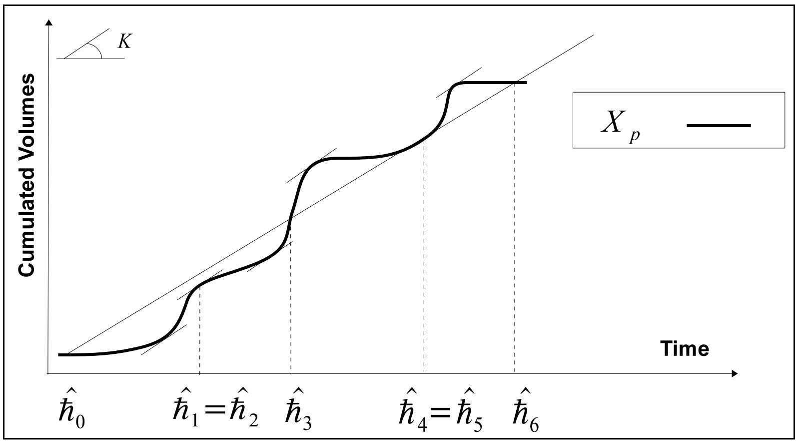


FIGURE 5 Derivation of would-be critical arrival times

Step II: Properties of $\hat{\hbar}_i(h_0)$

Let us show that the functions $\hat{\hbar}_i(h_0)$ are continuous and monotonic, in a way decreasing if i is odd meaning an off-peak P_i or increasing if i is even meaning a peak P_i. In the case of even i, consider $\Delta(h_0, \hbar)$ on $]h_b, h_e[\times]p_{i-1}, p_i[$. This is a continuous function with partial derivatives with respect to $\hbar$ and h_0 as follows:

$$\Delta_{h0}(h_0, \hbar) = x_p(h) - K < 0 \text{ and } \Delta_{\hbar}(h_0, \hbar) = K - x_p(\hbar) > 0 .$$

Consequently the equation $\Delta(h, \hbar) = 0$ defines implicitly a function $\hat{\hbar}_i(h)$ which is continuous and increasing on an interval $]a;b[$, in such a way that $(a, \lim_a \hat{\hbar}_i)$ and $(b, \lim_b \hat{\hbar}_i)$ lie on the boundary of $]h_b, h_e[\times]p_{i-1}, p_i[$. Hence, $]a;b[$ is such that $a = h_b$ or $\lim_{h\to a} \hbar_i = p_{i-1}$ and $b = h_e$ or $\lim_{h\to a} \hbar_i = p_i$. Furthermore, for all $\hbar$ $\Delta_{\hbar}(h_0, \hbar) < 0$ for $h < a$ and $\Delta_{\hbar}(h_0, \hbar) < 0$ for $h > b$.

Lastly, prolongating each $\hat{\hbar}_i$ on $[h_b, h_e]$ by the process defined above is continuous.

The case when i is odd is similar.

Step III: Proof that $W_{be}(h_0)$ is a continuous and decreasing function

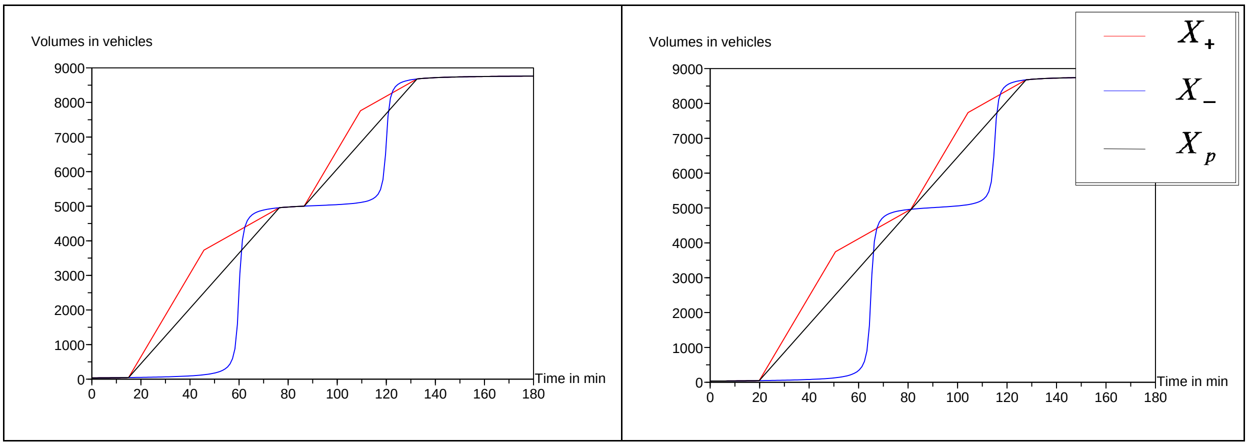
Derive $\hat{h}_i(h_0)$ and $\hat{w}_i(h_0)$ from $\hat{\hbar}_i(h_0)$ on the basis of (23) and (24). By straightforward substitution of (23) into (24) we get that

$$\hat{w}_{i+1} = \hat{w}_i + \frac{x_i^+ - K}{x_i^+}(\hat{\hbar}_{i+1} - \hat{\hbar}_i) \qquad (25)$$

As $\hat{\hbar}_{i+1} - \hat{\hbar}_i$ is a decreasing (resp. increasing) with respect to h_0 if i is even (resp. odd) hence $x_i^+ - K$ is positive (resp. negative) the incremental part in (25) is a decreasing function of h_0. Then each $\hat{w}_i$ is a decreasing function of h_0, owing to recursion and to the initial condition $W_0 = 0$. Concluding, the minimum W_{be} is a continuous and decreasing function of h_0 as the minimum of a sequence of such functions.

5. NUMERICAL EXPERIMENTS

Having implemented the algorithm in a computer program under the Scilab environment (*10*), a series of numerical experiments were performed by progressively moving two peak periods closer to each other (Figure 6). Initially there are two distinct queued periods, each of them with a single maximum of travel time. Then the two queues are merged into a single one with two maxima. Further, when the peak periods are close enough, the two maxima collapse into a single one yielding the same pattern as with a single peak period: the well-known pattern made up of one loading sub-period followed by an unloading one.



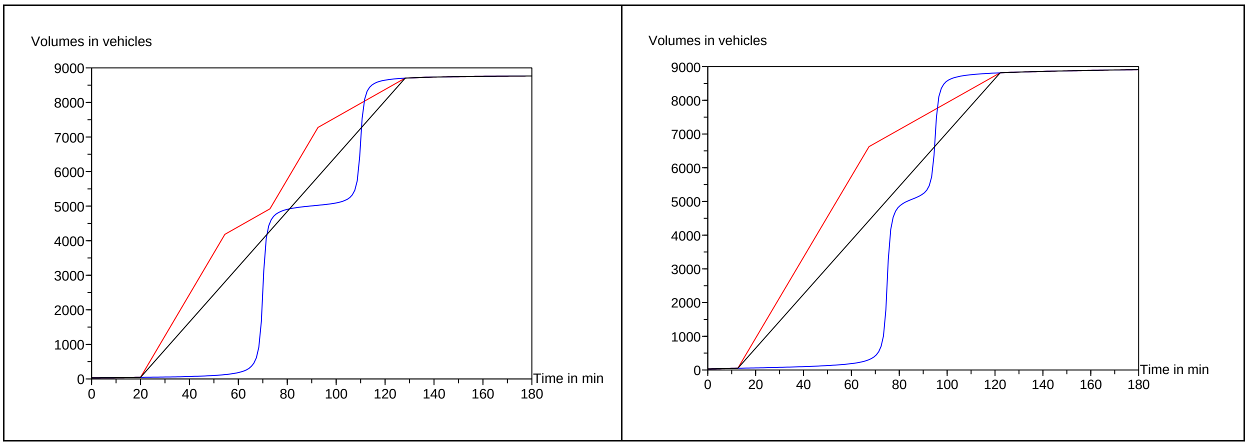


FIGURE 6 Some numerical experiments

6. CONCLUSION

This paper showed that relaxing the S-shape assumption on the pattern of preferred arrival times in the single bottleneck may give rise to a much more complex pattern of departure times, with potentially several queued periods and travel time maxima. Applications of such a model may include the assessment of transportation policies, such as congestion pricing or flextime promotion.

Among the improvements that would make sense, a major one is to introduce heterogeneity in the cost of schedule delay. Indeed complex road pricing schemes are based on the principle that one can segregate high schedule costs from lower ones by imposing time varying tolls. Therefore the heterogeneity in schedule delay cost functions and in the user cost of time is essential in assessing the benefits of such schemes.

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